

Distinguishing number and distinguishing index of natural and fractional powers of graphs

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Abstract

The distinguishing number (index) $D(G)$ ($D'(G)$) of a graph G is the least integer d such that G has an vertex labeling (edge labeling) with d labels that is preserved only by a trivial automorphism. For any $n \in \mathbb{N}$, the n -subdivision of G is a simple graph $G^{\frac{1}{n}}$ which is constructed by replacing each edge of G with a path of length n . The m^{th} power of G , is a graph with same set of vertices of G and an edge between two vertices if and only if there is a path of length at most m between them. The fractional power of G , denoted by $G^{\frac{m}{n}}$ is m^{th} power of the n -subdivision of G or n -subdivision of m -th power of G . In this paper we study the distinguishing number and distinguishing index of natural and fractional powers of G . We show that the natural powers more than two of a graph distinguished by three edge labels. Also we show that for a connected graph G of order $n \geq 3$ with maximum degree $\Delta(G)$, $D(G^{\frac{1}{k}}) \leq \min\{s : 2^k + \sum_{n=3}^s n^{k-1} \geq \Delta(G)\}$ and for $m \geq 3$, $D'(G^{\frac{m}{k}}) \leq 3$.

Keywords: Distinguishing index; Distinguishing number; Fractional power.

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1 Introduction

Let $G = (V, E)$ be a simple graph with n vertices. We use the standard graph notation ([7]). An automorphism of G is a permutation σ of the vertex set of G with the property that, for any vertices u and v , we have $u\sigma \sim v\sigma$ if and only if $u \sim v$ (note that $v\sigma$ denotes the image of the vertex v under the permutation σ). The set of all automorphisms of G , with the operation of composition of permutations, is a permutation group on V and is denoted by $Aut(G)$. A labeling of G , $\phi : V \rightarrow \{1, 2, \dots, r\}$, is r -distinguishing, if no non-trivial automorphism of G preserves all of the vertex labels. In other words, ϕ is r -distinguishing if for every non-trivial $\sigma \in Aut(G)$, there exists x in V such that

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$\phi(x) \neq \phi(x\sigma)$. The distinguishing number of a graph G has defined by Albertson and Collins [3] and is the minimum number r such that G has a labeling that is r -distinguishing. Similar to this definition, Kalinkowski and Pilśniak [11] have defined the distinguishing index $D'(G)$ of G which is the least integer d such that G has an edge colouring with d colours that is preserved only by a trivial automorphism. These indices has developed and number of papers published on this subject (see, for example [4, 14, 16]).

If x and y are two vertices of G , then the distance $d(x, y)$ between x and y , is defined as the length of a minimum path connecting x and y . The eccentricity of a vertex x is $ecc(x) = \max\{d(x, u) : u \in V(G)\}$ and the radius r and the diameter d of G are defined as the minimum and maximum eccentricity among vertices of G , respectively. A vertex u of G is called the central vertex if $ecc(u) = r$. The set of all central vertices of G , denoted by $Z(G)$, is called the center of G . For $k \in \mathbb{N}$, the k -power of G , denoted by G^k , is defined on the vertex set $V(G)$ by adding edges joining any two distinct vertices x and y with distance at most k [1, 15]. In other words, $E(G^k) = \{xy : 1 \leq d_G(x, y) \leq k\}$. Also the k -subdivision of G , denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge $v_i v_j$ of G with a path of length k , say $P_{v_i v_j}$. These k -paths are called superedges and any new vertex is an internal vertex, and is denoted by $w_l^{\{v_i, v_j\}}$ if it belongs to the superedge $P_{v_i v_j}$, $i < j$ and has distance l from the vertex v_i , where $l \in \{1, 2, \dots, k-1\}$. Note that for $k = 1$, we have $G^{1/1} = G^1 = G$, and if the graph G has v vertices and e edges, then the graph $G^{\frac{1}{k}}$ has $v + (k-1)e$ vertices and ke edges. The fractional power of G , denoted by $G^{\frac{m}{n}}$ is m^{th} power of the n -subdivision of G or n -subdivision of m -th power of G ([9]). Note that the graphs $(G^{\frac{1}{n}})^m$ and $(G^m)^{\frac{1}{n}}$ are different graphs. The fractional power of a graph has introduced by Iradmusa in [9]. He has investigated the chromatic number and clique number of fractional power of graphs. Also, he has studied domination number and independent domination number of fractional powers of graphs ([10]). Effantin and Kheddouci have studied the b -chromatic number of natural power of graphs and have obtained the exact value for the b -chromatic number of power graphs of a path and have determined bounds for the b -chromatic number of power graphs of a cycle ([6]). In the study of distinguishing number and distinguishing index of graphs, this naturally raises the question: What happens to the distinguishing number and the distinguishing index, when we consider the natural power and the fractional power of a graph? In this paper we would like to answer to this question. As usual we denote the complete graph, path and cycle of order n by K_n , P_n and C_n , respectively. Also $K_{1,n}$ is the star graph with $n + 1$ vertices.

In the next section, we state some results on the automorphism group and the distinguishing number and index of natural power of a graph and then compute theses two parameters for natural powers of paths and cycles. We show that the natural powers more than two of a graph distinguished by three edge labels. In Section 3 and 4, we study the distinguishing number and the distinguishing index of the fractional powers of graphs, respectively.

2 The distinguishing number and the distinguishing index of the natural powers of a graph

In this section, we consider the natural powers of a graph and study their distinguishing number and distinguishing index. We begin with the following lemma which follows from the definition of the power of graph.

Lemma 2.1 *Let G be a connected graph of order n and diameter d . Then*

- (i) *For every natural number $t \geq d$, $G^t = K_n$.*
- (ii) *(Theorem 1 in [8]) Let $k = mn$, where m and n are positive integers. Then $G^k = (G^m)^n$.*
- (iii) *Let x and y be two vertices of G with distance $d_G(x, y) = kq + r$ where $0 \leq r < k$. Then $d_{G^k}(x, y) = q + r$.*

Theorem 2.2 *Let G be a connected graph with radius r . Then*

- (i) *The automorphism group of G , $Aut(G)$, is a subgroup of $Aut(G^k)$ for $k \geq 2$.*
- (ii) *The automorphism group of G^{2t-1} , $Aut(G^{2t-1})$, is a subgroup of $Aut(G^{2t})$ for $1 \leq t \leq r$.*

Proof. (i) Since $Aut(G)$ is a group, it suffices to show that $Aut(G) \subseteq Aut(G^k)$. Let f be an automorphism of G . It is clear that v_i and v_j are adjacent in G^k if and only if $d_G(v_i, v_j) \leq k$ and this is true if and only if $d_G(f(v_i), f(v_j)) \leq k$ and so $f(v_i)$ and $f(v_j)$ are adjacent in G^k . So $f \in Aut(G^k)$, and the result follows.

(ii) If $t = 1$, then the result follows from Part (i). So let $t > 1$ and f be an automorphism of G^{2t-1} . It is clear that v_i and v_j are adjacent in G^{2t} if and only if $d_G(v_i, v_j) \leq 2t$ and this is true if and only if $d_{G^{2t-1}}(v_i, v_j) \leq 2$, and again this is true if and only if $d_{G^{2t-1}}(f(v_i), f(v_j)) \leq 2$ and so $d_G(f(v_i), f(v_j)) \leq 2t$ and hence $f(v_i)$ and $f(v_j)$ are adjacent in G^{2t} . So $f \in Aut(G^{2t})$, and the result follows. \square

By Parts (i) and (ii) of Theorem 2.2, we have the following results which are comparison between the distinguishing number of a graph and the distinguishing number of its natural powers:

Corollary 2.3 *Let G be a connected graph with radius r . Then*

- (i) *For every $k \geq 2$, $D(G) \leq D(G^k)$.*
- (ii) *For every $1 \leq t \leq r$, $D(G^{2t-1}) \leq D(G^{2t})$.*

Theorem 2.4 *Let G be a connected graph of order n with diameter d and radius r . If $Z(G) = \{x_1, \dots, x_t\}$, $t \geq 1$, is the center of G , then for $0 \leq i \leq d - r$ we have*

$$D(G^{r+i}) \geq |\{x \in V(G) \mid 0 \leq d_G(x_j, x) \leq i \text{ for some } j = 1, \dots, t\}|.$$

Proof. We first prove the case $i = 0$. If $i = 0$, then we have

$$\{x \in V(G) \mid d_G(x_j, x) = 0 \text{ for some } j = 1, \dots, t\} = \{x_1, \dots, x_t\}.$$

By definition of central vertex and power graph, the vertices x_1, \dots, x_t are the only vertices of G^r such that $\deg_{G^r} x_j = n - 1$ where $1 \leq j \leq t$. So the maps that fix non-central vertices and act on central vertices as a permutation of \mathbb{S}_t , are automorphisms of G^r . Because they preserve the adjacency relation in G^r . Thus we should have at least t labels so that we have a vertex distinguishing labeling that is not preserved by last automorphisms. Therefore $D(G^r) \geq t$.

For $i > 0$, the proof is similar. Indeed, the elements of the set $\{x \in V(G) \mid 0 \leq d_G(x_j, x) \leq i \text{ for some } j = 1, \dots, t\}$ are the only vertices of $V(G)$ such that their degree is $n - 1$ in G^r . \square

A graph G is Hamiltonian connected, if and only if every two distinct vertices of G are joined by a Hamiltonian path in G ([5],[18]). The following theorem implies that the cube of every connected graph is Hamiltonian connected (see also [13]).

Theorem 2.5 [19] *If G is a connected finite graph, then G^3 is Hamiltonian connected.*

We recall that a traceable graph is a graph that possesses a Hamiltonian path.

Theorem 2.6 [17] *If G is a traceable graph of order $n \geq 7$, then $D'(G) \leq 2$.*

The assumption $n \geq 7$ is substantial in this theorem, because $D'(K_{3,3}) = 3$. The following corollary shows that the natural powers more than two of a graph of order at least seven can be distinguished by two labels.

Corollary 2.7 *If G is a connected finite graph of order $n \geq 7$, then for any $i \geq 3$, $D'(G^i) \leq 2$.*

Proof. It can follow from Theorem 2.5 that G^i , $i \geq 3$, is Hamiltonian connected, and so it is a traceable graph. Since the order of graph is $n \geq 7$, so by Theorem 2.6 we have $D'(G^i) \leq 2$. \square

Remark 2.8 *If G is a connected finite graph of order $1 \leq n \leq 5$ and G is not a path graph, then the diameter of G is less than or equal to three. So we have $D'(G^i) = D'(K_n) = 3$ for $i \geq 3$. For $n = 6$, using table of graphs, observe that the diameter of G is less than or equal to 3 except eight cases. In that eight cases the diameter of G is 4 (G is not a path graph), and so $D'(G^i) = D'(K_6) = 2$ for $i \geq 4$. It can be easily computed that $D'(G^3) \leq 3$ for that eight cases.*

Motivated by Corollary 2.7 and Remark 2.8, we shall prove that the natural powers more than two of any graph can be distinguished by three edge labels. To do this, we need to consider the distinguishing index (and distinguishing number) of natural powers of paths.

Theorem 2.9 *Let $n \geq 4$ and $k \geq 2$ be integers. The distinguishing number of the path P_n^k with diameter d and radius r is as follows:*

$$D(P_n^k) = \begin{cases} 2 & 1 \leq k \leq r, \\ 2k - n & r + 1 \leq k \leq d. \end{cases}$$

Proof. It can be seen that the degree sequence of P_n^k for $1 \leq k \leq r - 1$ is as follows (note that the i -th term of the degree sequence is degree of the i -th vertex of P_n from left side and the vertices of P_n is denoted by x_1, \dots, x_n):

$$\{deg_{P_n^k} x_i\}_{i=1}^n = \{k, k+1, \dots, k+(k-1), \underbrace{2k, \dots, 2k}_{(n-2k)\text{-times}}, k+(k-1), \dots, k+1, k\}.$$

Also the degree sequence of P_n^r is as follows

$$\{deg_{P_n^r} x_i\}_{i=1}^n = \begin{cases} \{r, r+1, \dots, n-1, n, n-1, \dots, r+1, r\} & n \text{ is even,} \\ \{r, r+1, \dots, n-1, n, n, n-1, \dots, r+1, r\} & n \text{ is odd.} \end{cases}$$

For P_n^k , $1 \leq k \leq r$, by assigning the two vertices of degree i , $k \leq i \leq 2k$, the labels 1 and 2, and assigning the remaining vertices the label 1 we have a distinguishing labeling. Because if f is an automorphism of P_n^k such that fixes all the two vertices with degree i , $k \leq i \leq 2k - 1$, then f fixes all vertices. Since we used two labels, $D(P_n^k) = 2$ for $1 \leq k \leq r$. Also, by considering the adjacency relation we have $Aut(P_n^k) = Aut(P_n)$ for $1 \leq k \leq r$.

The degree sequence of P_n^k for $r + 1 \leq k \leq d$ is as follows:

$$\{deg_{P_n^k} x_i\}_{i=1}^n = \{k, k+1, \dots, n-1, \underbrace{n, \dots, n}_{(2k-n)\text{-times}}, n-1, \dots, k+1, k\}.$$

By assigning all the two vertices of degree i , $k \leq i \leq n - 1$, the labels 1 and 2, and assigning all vertices of degree n , the distinguishing labeling of a complete graph with order of the number of this kind of vertices, we have $D(P_n^k) = 2k - n$ where $r + 1 \leq k \leq d$. Also, by considering the adjacency relation and the number of the vertices of induced complete subgraph we can obtain that $|Aut(P_n^k)| = 2|Aut(K_{2k-n})|$ where $r + 1 \leq k \leq d$. \square

Using the proof of Theorem 2.9, we have the following result which is the distinguishing index of natural powers of paths.

Corollary 2.10 *Let $n \geq 3$ and $k \geq 2$ be integers. The distinguishing number of the path P_n^k with diameter d and radius r is as follows:*

$$D'(P_n^k) = \begin{cases} D'(P_n) & 1 \leq k \leq r, \\ D'(K_{2k-n-2}) & r + 1 \leq k \leq d. \end{cases}$$

Now, we are ready to state the following result which obtain from Corollary 2.7, Remark 2.8 and Corollary 2.10. This result implies that all natural powers more than two of a graph G distinguished by three edge labels.

Corollary 2.11 *If G is a connected finite graph of order n , then $D'(G^m) \leq 3$ for $m \geq 3$.*

Corollary 2.11 is true for the powers more than two of an arbitrary graph. To see what happen to G^2 , we need some results. The following Lemma is an easy exercise in graph theory literature.

Lemma 2.12 *Let G be a connected graph of order n . If $|E(G)| \geq \binom{n-1}{2} + 2$, then G has a Hamiltonian cycle.*

Theorem 2.13 [2] *If G^2 is not a complete graph, then the number of edges in which were added to G to construct G^2 is at least $n - 2$.*

Corollary 2.14 *Let G be a connected graph of order $n \geq 7$ such that G^2 is not a complete graph. If $|E(G)| \geq \frac{1}{2}(n^2 + 10 - 5n)$, then $D'(G^2) \leq 2$.*

Proof. By Theorem 2.13 we have $|E(G^2)| \geq n - 2 + |E(G)|$. If $n - 2 + |E(G)| \geq \binom{n-1}{2} + 2$ then G^2 has a Hamilton cycle by Lemma 2.12. But $n - 2 + |E(G)| \geq \binom{n-1}{2} + 2$ concludes that $|E(G)| \geq \frac{1}{2}(n^2 + 10 - 5n)$. Now by Corollary 2.7 we have the result. \square

Before ending this section, let to consider the natural power of a cycle graph of order n , i.e., C_n , and study its distinguishing number and index. The following theorem is about the automorphism group of powers of a cycle of order n , i.e., C_n :

Theorem 2.15 *Let $n \geq 3$ and $k \geq 2$ be integers. Then*

$$Aut(C_n^k) = \begin{cases} Aut(C_n) & n > 2k, \\ Aut(K_n) & n \leq 2k. \end{cases}$$

Proof. Let $V(C_n) = \{x_1, \dots, x_n\}$ and $E(C_n) = \{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_n, x_1\}\}$. Let $n > 2k$, we shall prove that $Aut(C_n^k) = Aut(C_n^{k-1})$. Let $f \in Aut(C_n^k)$ and $f \notin Aut(C_n^{k-1})$. Suppose that there exist $x_s, x_t \in V(C_n)$ such that $f(x_s) = x_t$. Since $f \notin Aut(C_n^{k-1})$, so either there is i , $1 \leq i \leq k - 1$ such that $f(x_{s+i}) = x_{t+k}$, or $f(x_{s-i}) = x_{t+k}$. Without loss of generality we can assume that $f(x_{s+i}) = x_{t+k}$. Observe that the distinct vertices

$$x_{s-(k-i)}, x_{s-(k-i-1)}, \dots, x_{s-2}, x_{s-1}, x_{s+1}, x_{s+2}, \dots, x_{s+i-1}, x_{s+i+1}, \dots, x_{s+k}$$

are adjacent to both of x_s and x_{s+i} in C_n^k . Since the automorphisms preserve the adjacency, so the image of these vertices should be adjacent to both of x_t and x_{t+k} . On

the other hand $x_{t+1}, x_{t+2}, \dots, x_{t+k-1}$ are the only vertices of C_n^k that are adjacent to both of x_t and x_{t+k} . Thus we have

$$\{f(x_{s+i-k}), f(x_{s+i-k+1}), \dots, f(x_s), f(x_{s+1}), \dots, f(x_{s+k})\} = \{x_t, x_{t+1}, \dots, x_{t+k}\}.$$

But these two sets have the distinct cardinality, which is a contradiction. So we have $\text{Aut}(C_n^k) = \text{Aut}(C_n^{k-1})$, and therefore $\text{Aut}(C_n^k) = \text{Aut}(C_n)$. For the case $n \leq 2k$, we have $C_n^k = K_n$. Therefore $\text{Aut}(C_n^k) = \text{Aut}(K_n)$. \square

The following corollary can be verified directly from Theorem 2.15.

Corollary 2.16 *Let $n \geq 3$ and $k \geq 2$ be integers. We have*

$$(i) \quad D(C_n^k) = \begin{cases} D(C_n) & n > 2k, \\ D(K_n) & n \leq 2k. \end{cases}$$

$$(ii) \quad D'(C_n^k) = \begin{cases} D'(C_n) & n > 2k, \\ D'(K_n) & n \leq 2k. \end{cases}$$

The characterization of graphs G with $\text{Aut}(G) = \text{Aut}(G^2)$, is an interesting problem, and we think that most of graphs have this property. However, until now all attempts to characterize these graphs failed, and it remains as open problem. Let to end this section by posing the following conjecture:

Conjecture 2.17 *(i) If G is a connected graph with diameter d and radius r such that $r < d \leq 2r - 2$, then $\text{Aut}(G) = \text{Aut}(G^2)$.*

(ii) If G is a connected bipartite graph with radius $r > 2$, then $\text{Aut}(G) = \text{Aut}(G^2)$.

3 Distinguishing number of the fractional power of graphs

In this section, we study the distinguishing number of the fractional powers of graphs. It can easily be verified that for $n \geq 2$ and $k \geq 2$, $D(P_n^{\frac{1}{k}}) = 2$. Also for $n \geq 3$ and $k \geq 2$, $D(C_n^{\frac{1}{k}}) = 2$. We state and prove the following lemma to study more on the distinguishing number of the fractional power of graphs.

Lemma 3.1 *Let G be a connected graph of order $n \geq 3$ which is not a cycle. If $f \in \text{Aut}(G^{\frac{1}{k}})$, then $f|_{V(G)} \in \text{Aut}(G)$.*

Proof. Let $\{v_1, \dots, v_n\}$ be the vertex set of G . Suppose that $w_t^{\{v_i, v_j\}}$, $1 \leq t \leq k-1$ is an internal vertex of $G^{\frac{1}{k}}$ such that $f(w_t^{\{v_i, v_j\}}) = v_s$ for some $s \in \{1, \dots, n\}$. Since $\deg_{G^{\frac{1}{k}}}(w_t^{\{v_i, v_j\}}) = 2$, so $\deg(v_s) = 2$ (note that $\deg_G(v_i) = \deg_{G^{\frac{1}{k}}}(v_i)$ for all $i \in \{1, \dots, n\}$). Let v_{s_1} be the adjacent vertex to v_s in G and $w_1^{\{v_s, v_{s_1}\}}$ be the adjacent vertex to v_s in $G^{\frac{1}{k}}$. Without loss of generality we can assume that

$$f(w_{t-1}^{\{v_i, v_j\}}) = w_1^{\{v_s, v_{s_1}\}}, f(w_t^{\{v_i, v_j\}}) = w_2^{\{v_s, v_{s_1}\}}, \dots, f(w_1^{\{v_i, v_j\}}) = w_{t-1}^{\{v_s, v_{s_1}\}}, f(v_i) = w_t^{\{v_s, v_{s_1}\}}.$$

Then $\deg_G(v_i) = 2$. Continuing this process, we see that any vertex of G has degree two, and so G is a cycle, which is a contradiction. \square

Observation 3.2 *Let G be a connected graph of order $n \geq 3$ which is not a cycle. Let $i < j$ and v_i and v_j be two adjacent vertices of G . Suppose that f is an automorphism of $G^{\frac{1}{k}}$ such that $f(v_i) = v_{i'}$ and $f(v_j) = v_{j'}$. We have two following cases:*

Case 1) *If $i' < j'$, then $f(w_t^{\{v_i, v_j\}}) = w_t^{\{v_{i'}, v_{j'}\}}$, where $1 \leq t \leq k-1$.*

Case 2) *If $i' > j'$, then $f(w_t^{\{v_i, v_j\}}) = w_{k-t}^{\{v_{i'}, v_{j'}\}}$, where $1 \leq t \leq k-1$.*

Corollary 3.3 *Let G be a connected graph of order $n \geq 3$ which is not a cycle. Then for every natural number k ,*

$$(i) |Aut(G^{\frac{1}{k}})| = |Aut(G)|.$$

$$(ii) D(G^{\frac{1}{k}}) \leq D(G).$$

Proof. (i) It follows from Observation 3.2.

(ii) By Observation 3.2, if we label the vertices of the graph G with $D(G)$ labels in a distinguishing way and assign the internal vertices the label 1, then we have a distinguishing labeling. Therefore $D(G^{\frac{1}{k}}) \leq D(G)$. \square

We need the following definition to obtain more results on the distinguishing number of fractional powers of graphs:

Definition 3.4 [12] *The total distinguishing number $D''(G)$ of a graph G is the least number d such that G has a total colouring with d colours that is preserved only by the identity automorphism of G .*

We also need the following theorem which gives an upper bound for the total distinguishing number of a graph:

Theorem 3.5 [12] *If G is a connected graph of order $n \geq 3$ with maximum degree $\Delta(G)$, then $D''(G) \leq \lceil \sqrt{\Delta(G)} \rceil$.*

Note that this bound is sharp, because $D''(K_{1,n}) = \lceil \sqrt{n} \rceil$.

Theorem 3.6 *If G is a connected graph of order $n \geq 3$, then $D(G^{\frac{1}{2k}}) = D''(G^{\frac{1}{k}})$.*

Proof. It suffices to note that $G^{\frac{1}{2k}}$ is constructed by replacing each edge $v_i v_j$ of $G^{\frac{1}{k}}$ with a path of length 2, say $P_{v_i v_j}$. So if we consider the label of the edges in total labeling of $G^{\frac{1}{k}}$ as the label of internal vertices of $G^{\frac{1}{2k}}$ then the result follows. \square

By Theorems 3.5 and 3.6 we have the following corollary:

Corollary 3.7 *If G is a connected graph of order $n \geq 3$ with maximum degree $\Delta(G)$, then $D(G^{\frac{1}{2}}) \leq \lceil \sqrt{\Delta(G)} \rceil$.*

Now we want to obtain a better upper bound for $D(G^{\frac{1}{k}})$. For this purpose, let $S_k^G(x)$, $k \geq 0$ denote a sphere of radius k with a center x , i.e., the set of all vertices of G at distance k from x .

Theorem 3.8 *If G is a connected graph of order $n \geq 3$ with maximum degree $\Delta(G)$, then $D(G^{\frac{1}{k}}) \leq \min\{s : 2^k + \sum_{n=3}^s n^{k-1} \geq \Delta(G)\}$.*

Proof. The basic idea follows from the proof of Theorem 3.5 (see [12]). If G is a cycle of order n , then since for $n \geq 3$ and $k \geq 2$, $D(C_n^{\frac{1}{k}}) = 2$ we have the result. Let v_0 be a vertex of degree $\Delta(G)$ and $G \neq K_{1,n-1}$ (the case $G = K_{1,n-1}$ has been considered in next theorem). So $S_2^G(v_0)$ is nonempty. We label v_0 with the label 2 and consider a BFS tree T of G rooted at v_0 . We will first label the vertices of the tree $T^{\frac{1}{k}}$. For a given vertex v of G , denote $M_t(v) = \{(w_1^{\{v,u\}}, \dots, w_{k-1}^{\{v,u\}}, u) : vu \in E(G)\}$ (note that the vertices $w_1^{\{v,u\}}, \dots, w_{k-1}^{\{v,u\}}$ are internal vertices on P_{vu}). Let $S_1^G(v_0) = \{v_1, \dots, v_p\}$. Without loss of generality we can assume that v_1 has a neighbour in $S_2^G(v_0)$. We label both k -ary $(w_1^{\{v_0,v_1\}}, \dots, w_{k-1}^{\{v_0,v_1\}}, v_1)$ and $(w_1^{\{v_0,v_2\}}, \dots, w_{k-1}^{\{v_0,v_2\}}, v_2)$ with a k -ary $(1, \dots, 1)$. Then we label each k -ary of $M_t(v_0) \setminus \{(w_1^{\{v_0,v_1\}}, \dots, w_{k-1}^{\{v_0,v_1\}}, v_1), (w_1^{\{v_0,v_2\}}, \dots, w_{k-1}^{\{v_0,v_2\}}, v_2)\}$ with a distinct k -ary of labels different from $(1, \dots, 1)$. Thus $(1, \dots, 1)$ appears twice as a k -ary of labels in $M_t(v_0)$. We will then label the vertices of graph G in such a way that v_0 will be the only vertex of G labeled with the label 2 such that k -ary $(1, \dots, 1)$ appears twice in the $M_t(v_0)$. Hence v_0 will be fixed by every automorphism of $G^{\frac{1}{k}}$ preserving the labeling. Therefore all vertices in $S_1^G(v_0)$ and all internal vertices on $P_{v_0v_1}, \dots, P_{v_0v_p}$ will also be fixed, except, possibly $(w_1^{\{v_0,v_1\}}, \dots, w_{k-1}^{\{v_0,v_1\}}, v_1)$ and $(w_1^{\{v_0,v_2\}}, \dots, w_{k-1}^{\{v_0,v_2\}}, v_2)$. To distinguish them we label the sets $\{(w_1^{\{v_1,u\}}, \dots, w_{k-1}^{\{v_1,u\}}, u) \in M_t(v_1) : u \in S_2^G(v_0)\}$ and $\{(w_1^{\{v_2,u\}}, \dots, w_{k-1}^{\{v_2,u\}}, u) \in M_t(v_2) : v_2u \in E(T), u \in S_2^G(v_0)\}$ with two distinct sets of k -ary of labels (this is possible because each of these sets contains at most $\Delta(G) - 1$ elements, and we have $\Delta(G)$ distinct k -ary of labels). Therefore every internal vertex on the superedges $P_{v_0v_1}, \dots, P_{v_0v_p}$ and P_{v_1u}, P_{v_2u} where $u \in S_2^G(v_0)$ will be fixed by every automorphism of $G^{\frac{1}{k}}$ preserving the presented labeling. For $i = 3, \dots, p$, we then label all elements of $\{(w_1^{\{v_i,u\}}, \dots, w_{k-1}^{\{v_i,u\}}, u) : v_iu \in E(T), u \in S_2^G(v_0)\}$ with distinct k -ary of labels different from the $(1, \dots, 1)$. This is possible. Thus all other vertices in $S_2^G(v_0)$ and all internal vertices on $P_{v_0v_1}, \dots, P_{v_0v_p}$ and $P_{v_1u}, \dots, P_{v_pu}$ where $u \in S_2^G(v_0) \setminus S_1^G(v_0)$ will be also fixed.

Then we proceed recursively with respect to the radius k of subsequent sphere $S_k^G(v_0)$ according to the ordering of the BFS tree T . Suppose all vertices of $S_i^G(v_0) = \{u_1, \dots, u_{l_i}\}$, $i = 0, 1, \dots, k$ and all internal vertices on $P_{v_a v_b}$, where $v_a, v_b \in S_i^G(v_0)$, are fixed by every automorphism of $G^{\frac{1}{k}}$ preserving labels. For each subsequent vertex u_j , $j = 1, \dots, l_k$ we label every k -ary $(w_1^{\{u_j,u\}}, \dots, w_{k-1}^{\{u_j,u\}}, u)$ where u is a descendent of u_j in T , with a distinct k -ary of labels except for $(1, \dots, 1)$. This is possible because the number of k -ary to be labeled is not greater than the number of admissible k -ary of labels. Thus all neighbours of u_j in $S_{k+1}^G(v_0)$ and all internal vertices on the superedges $P_{u_j u}$, where u is a descendent of u_j in T , will be also fixed.

Finally, we label all remaining vertex in $V(G^{\frac{1}{k}}) \setminus V(T^{\frac{1}{k}})$ with the label 2. It is easy to see that if v is a vertex labeled with the label 2 such that the k -ary $(1, \dots, 1)$ appears twice in $M_t(v)$, then $v = v_0$. Hence all vertices of $G^{\frac{1}{k}}$ are fixed by any automorphism of $G^{\frac{1}{k}}$ preserving this labeling. \square

Now we shall show that the inequality of Theorem 3.8 is sharp.

Theorem 3.9 *For $m \geq 3$ and $k \geq 2$, $D(K_{1,m}^{\frac{1}{k}}) = \min\{s : 2^k + \sum_{n=3}^s n^{k-1} \geq m\}$.*

Proof. Let to denote the one vertex in a part of $K_{1,m}$ by v_0 and the remaining vertices by v_1, \dots, v_m . So the label of v_0 can be arbitrary (because it is the only vertex of degree m). Using $\min\{s : 2^k + \sum_{n=3}^s n^{k-1} \geq m\}$ labels, we have at least m different k -ary (c_1, \dots, c_k) of labels. If we assign these m different k -ary (c_1, \dots, c_k) to the vertex set of m superedges that replaced with the m edges v_0v_1, \dots, v_0v_m (except the vertex v_0), then we have distinguishing labeling. If we use less than $\min\{s : 2^k + \sum_{n=3}^s n^{k-1} \geq m\}$ labels then we have less than m different k -ary of labels, so there would exist at least two paths (start with v_0 and end with v_i and v_j) labeled similarly. Thus we can find a non-trivial automorphism preserving such a labeling. \square

4 Distinguishing index of the fractional powers of graphs

In this section, we study the distinguishing index of the fractional powers of graphs. It can be easily verified that for $n \geq 2$ and $k \geq 2$, $D'(P_n^{\frac{1}{k}}) = 2$ and for $n \geq 3$ and $k \geq 2$, $D'(C_n^{\frac{1}{k}}) = 2$. We begin with the following theorem.

Theorem 4.1 *Let G be a connected graph of order $n \geq 3$ such that it is not a cycle. For any $k \geq 2$, $D(G^{\frac{1}{k+1}}) \leq D'(G^{\frac{1}{k}})$.*

Proof. We define a distinguishing vertex labeling for $G^{\frac{1}{k+1}}$ with $D'(G^{\frac{1}{k}})$ labels. Suppose that $P_{v_i v_j}$ is a superedge that has been replaced with the edge $v_i v_j$ in structure of $G^{\frac{1}{k}}$. If we assign the internal vertices have been lied on corresponding superedge in construction of $G^{\frac{1}{k+1}}$, the label of edges $P_{v_i v_j}$ in $G^{\frac{1}{k}}$ and assign the remaining vertices the label 1, then by Observation 3.2 we can conclude that the labeling is distinguishing. Since we used $D'(G^{\frac{1}{k}})$ labels, the result follows. \square

Theorem 4.2 *Let G be a connected graph of order $n \geq 3$ such that it is not a cycle. Then $D'(G^{\frac{1}{2}}) \leq \lceil \frac{-1 + \sqrt{1 + 8D'(G)}}{2} \rceil$.*

Proof. First using the label of edges, we partition the edge set of G . So we have $D'(G)$ classes, each class contains the edges with similar labels. The elements of $[i]$ -th class are denoted by e_{i1}, \dots, e_{is_i} , where $1 \leq i \leq D'(G)$ and $\sum_{i=1}^{D'(G)} s_i = |E(G)|$. We know

that each edge of G is replaced with a path of length 2 in $G^{\frac{1}{2}}$. Let the edge e_{ij} in G be replaced with two edges e_{ij}^1 and e_{ij}^2 in $G^{\frac{1}{2}}$. For $1 \leq i \leq D'(G)$, we assign the distinct pairs (c_{i1}, c_{i2}) of labels to all new edges e_{ij}^1 and e_{ij}^2 , where $1 \leq j \leq s_i$ such that

- (i) $c_{i1} \neq c_{i2}$ for $1 \leq i \leq D'(G)$.
- (ii) $\{c_{i1}, c_{i2}\} \neq \{c_{i'1}, c_{i'2}\}$ for $1 \leq i, i' \leq D'(G)$ and $i \neq i'$.

By Observation 3.2 this labeling is distinguishing. Since the number of labels that have been used, is $\min\{s : \sum_{i=1}^s i \geq D'(G)\}$, so $D'(G^{\frac{1}{2}}) \leq \min\{s : \sum_{i=1}^s i \geq D'(G)\}$. By an easy computation, we see that

$$\min\{s : \sum_{i=1}^s i \geq D'(G)\} = \lceil \frac{-1 + \sqrt{1 + 8D'(G)}}{2} \rceil.$$

Therefore we have the result. \square

Let (c_1, \dots, c_k) be an k -ary of labels such that it is not symmetric, i.e., there exists i , $1 \leq i \leq k$ such that $c_i \neq c_{k-i}$. Let the minimum number of labels that have been used in construction of $D'(G)$ numbers of such k -ary, is d'_k . Then we have the following theorem.

Theorem 4.3 *Let G be a connected graph of order $n \geq 3$ such that it is not a cycle. Then $D'(G^{\frac{1}{k}}) \leq d'_k$.*

Proof. First, we partition the edge set of G using the label of edges. So we have $D'(G)$ classes, each class contains the edges with similar labels. The elements of $[i]$ -th class are denoted by e_{i1}, \dots, e_{is_i} , where $1 \leq i \leq D'(G)$ and $\sum_{i=1}^{D'(G)} s_i = |E(G)|$. We know that each edge of G is replaced with a path of length k in $G^{\frac{1}{k}}$. Let the edge e_{ij} in G be replaced with two edges $e_{ij}^1, \dots, e_{ij}^k$ in $G^{\frac{1}{k}}$. For $1 \leq i \leq D'(G)$, we assign the above explained distinct k -ary (c_{i1}, \dots, c_{ik}) of labels to all new edges $e_{ij}^1, \dots, e_{ij}^k$, where $1 \leq j \leq S_i$. By Observation 3.2 this labeling is distinguishing. Since the number of labels that have been used, is d'_k , so we have the result. \square

By Theorems 3.8, 4.1 and 4.3, we have two following upper bound for $D(G^{\frac{1}{k}})$:

$$\begin{cases} D(G^{\frac{1}{k}}) \leq \lambda_1 := d'_{k-1}, \\ D(G^{\frac{1}{k}}) \leq \lambda_2 := \min\{s : 2^k + \sum_{n=3}^s n^{k-1} \geq \Delta(G)\}. \end{cases}$$

This raises the question “which upper bound is better”? It depends to graph. For instance, the upper bound λ_1 is better than λ_2 for fractional powers of star graphs, $K_{1,m}^{\frac{1}{k}}$ and the situation is different for the one half power of $F_2 = K_1 + 2K_2$, i.e., $F_2^{\frac{1}{2}}$.

Now by Corollary 2.11 and Theorem 4.3 we have the following result.

Corollary 4.4 *If G is a connected finite graph of order $n \geq 3$, then $D'(G^{\frac{m}{k}}) \leq 3$ for $m \geq 3$ and $k \geq 1$.*

Proof. We know that $G^{\frac{m}{k}}$ means $(G^{\frac{1}{k}})^m$ or $(G^m)^{\frac{1}{k}}$. In the case $(G^{\frac{1}{k}})^m$ the result follows directly from Corollary 2.11. For the case $(G^m)^{\frac{1}{k}}$ we have $D'((G^m)^{\frac{1}{k}}) \leq d'_k$ by Theorem 4.3 and d'_k is the minimum number of labels that have been used in construction of $D'(G^m)$ numbers of symmetric k -ary. Since $D'(G^m) \leq 3$, so $d'_k \leq 3$. Therefore we have $D'((G^m)^{\frac{1}{k}}) \leq 3$. \square

The following results follows easily from Corollaries 2.3 and 3.3.

Corollary 4.5 *If G is a connected finite graph of order $n \geq 3$, then for $m \geq 3$ and $k \geq 1$ we have*

$$(i) \ D(G^{\frac{1}{k}}) \leq D((G^{1/k})^m).$$

$$(ii) \ D((G^m)^{\frac{1}{k}}) \leq D(G^m).$$

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